

ANALYSING THE ELASTICITY DIFFERENCE TENSOR OF GENERAL RELATIVITY

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ABSTRACT. The elasticity difference tensor, used in [1] to describe elasticity properties of a continuous medium filling a space-time, is here analysed from the point of view of the space-time connection. Principal directions associated with this tensor are compared with eigendirections of the material metric. Examples concerning spherically symmetric and axially symmetric space-times are then presented.

1. INTRODUCTION

In recent years there has been a growing interest in the theory of general relativistic elasticity. Based on the classical Newtonian elasticity theory going back to the 17th century and Hooke's law, some authors began to adapt the theory of elasticity to the relativity due to the necessity to study many astrophysical problems as the interaction between the gravitational field and an elastic solid body in the description of stellar matter, as well as to understand the interaction of gravitational waves and gravitational radiation and to study deformations of neutron star crusts. One of the first elastic phenomenon considered in the relativistic context was Weber's observation of the elastic response of an aluminium cylinder to gravitational radiation and the detection of gravitational waves [2], [3] and [4]. Neutron stars have attracted attention since it has been argued [5] that the crusts of neutron stars are in elastic states and since it has been established the existence of a solid crust and speculated the possibility of solid cores in neutron stars, [6], [7], [8].

There were many attempts to formulate a relativistic version of elasticity theory. Thereby laws of non relativistic continuum mechanics had to be reformulated in a relativistic way. The study of elastic media in special relativity was firstly carried out by Noether [9] in 1910 and by Born [10], Herglotz [11] and Nordström [12] in 1911. The discussion of elasticity theory in general relativity started with Synge [13], De Witt [14], Rayner [15], Bennoun [16], [17], Hernandez [18] and Maugin [19] ¹. In 1973 Carter and Quintana [20] developed a relativistic formulation of the concept of a perfectly elastic solid and constructed a quasi-Hookean perfect elasticity theory suitable for applications to high-pressure neutron star matter. Recently, Karlovini and Samuelsson [1] gave an important contribution to this topic, extending the results of Carter and Quintana (see also [21], [22]). Other relevant formulations of elasticity in the framework of general relativity were given by Kijowski and Magli ([23], [24]) who presented a gauge-type theory of relativistic elastic media and a corresponding generalization [25]. The same authors also studied interior solutions of the Einstein field equations in elastic media ([26], [27]).

The recent increasing consideration of relativistic elasticity in the literature shows

¹Relativistic elasticity has been treated in the mid-20th century until the early seventies by many other authors. For further references, see, for example, [19], and for later references see also [23], [1].

the win of recognition and importance of this topic, motivating for a detailed study of quantities used in this context, the elasticity difference tensor defined in [1] being one of them. This tensor occurs in the relativistic Hadamard elasticity tensor and in the Euler equations for elastic matter. However, one can recognize the geometric role of the elasticity difference tensor, since, in principle, it can be used to understand the influence of the material metric (inheriting elastic properties) on the curvature of the space-time.

Here, in section 2, general results about relativistic elasticity are presented. In section 3, the elasticity difference tensor is analysed and principal directions associated with this tensor are compared with the eigendirections of the pulled-back material metric. A specific orthonormal tetrad is introduced to write a general form of the elasticity difference tensor, which brings in Ricci rotation coefficients used in the 1 + 3 formalism [28] and the linear particle densities.

Finally, in section 4, we apply the results obtained to a static spherically symmetric space-time and an axially symmetric non-rotating space-time. The software *Maple GRTensor* was used to perform some calculations.

2. GENERAL RESULTS

Let (M, g) be a space-time manifold, i.e. a 4-dimensional, paracompact, Hausdorff, smooth manifold endowed with a Lorentz metric g of signature $(-, +, +, +)$, U being a local chart around a point $p \in M$. Suppose that U is filled with a continuum material. The material space \mathcal{X} is an abstract 3-dimensional manifold, each point in \mathcal{X} representing an idealized particle of the material. Moreover, the space-time configuration of the material is described by a mapping

$$\Psi : U \subset M \longrightarrow \mathcal{X},$$

which associates to each point p of the space-time the particle \bar{p} of the material which coincides with p at a certain time. Therefore $\Psi^{-1}(\bar{p})$ represents the flowline of the particle \bar{p} . The operators push-forward Ψ_* and pull-back Ψ^* will be used to take contravariant tensors from M to \mathcal{X} and covariant tensors from \mathcal{X} to M , respectively, in the usual way.

If $\{\xi^A\}$ ($A = 1, 2, 3$) is a coordinate system in \mathcal{X} and $\{\omega^a\}$ ($a = 0, 1, 2, 3$)² a coordinate system in $U \subset M$, then the configuration of the material can be described by the fields $\xi^A = \xi^A(\omega^a)$. The mapping $\Psi_* : T_p M \longrightarrow T_{\Psi(p)} \mathcal{X}$ gives rise to a (3×4) matrix (the relativistic deformation gradient) whose entries are $\xi_a^A = \frac{\partial \xi^A}{\partial \omega^a}$. Assuming that the world-lines of the particles $\Psi^{-1}(\bar{p})$ are timelike, the relativistic deformation gradient is required to have maximal rank and the vector fields $u^a \in T_p M$, satisfying $u^a \xi_a^B = 0$, are required to be timelike and future oriented. The vector field u^a is the velocity field of the matter and its components obey $u^a u_a = -1$, $u^a \xi_a^B = 0$ and $u^0 > 0$, [23].

One needs to consider, in the material space \mathcal{X} , a Riemannian metric η_{AB} , describing the “rest frame” space distances between particles calculated in the “locally relaxed state” or in the “unsheared state” of the material and often taken as the material metric. These approaches are presented in [23] and in [1], respectively.

²Capital Latin indices A,B,... range from 1 to 3 and denote material indices. Small Latin indices a,b,... take the values 0,1,2,3 and denote space-time indices.

Let ϵ_{ABC} be the volume form of η_{AB} , with $\Psi^*\epsilon_{ABC} = \epsilon_{abc} = \epsilon_{abcd}u^d$. The particle density form is $n_{ABC} = n\epsilon_{ABC}$, with n the particle density yielding the number of particles in a volume of \mathcal{X} when integrated over that volume. One can define, see [1], a new tensor k_{AB} , which has n_{ABC} as its volume form and is conformal to η_{AB} : $k_{AB} = n^{\frac{2}{3}}\eta_{AB}$. This tensor will be taken as the *material metric* in \mathcal{X} . The pull-back of the material metric

$$k_{ab} = \Psi^*k_{AB} = \xi_a^A \xi_b^B k_{AB} \quad (1)$$

and the (usual) projection tensor

$$h_{ab} = g_{ab} + u_a u_b \quad (2)$$

are Riemannian metric tensors on the subspace of $T_p M$ orthogonal to u^a . These tensors are symmetric and satisfy $k_{ab}u^a = 0 = h_{ab}u^a$.

The state of strain of the material can be measured by the relativistic strain tensor, according to e.g. [26], [27]:

$$s_{ab} = \frac{1}{2}(h_{ab} - \eta_{ab}) = \frac{1}{2}(h_{ab} - n^{-\frac{2}{3}}k_{ab}). \quad (3)$$

This tensor is also named as constant volume shear tensor (see [20], [1]). The material is said to be “locally relaxed” at a particular point of space-time if the material metric and the projection tensor agree at that point, i.e. if the strain tensor vanishes.

When considering elastic matter sources in general relativity, one is confined to a stress-energy tensor taking the form $T_{ab} = -\rho g_{ab} + 2\frac{\partial \rho}{\partial g^{ab}} = \rho u_a u_b + p_{ab}$, where $p_{ab} = 2\frac{\partial \rho}{\partial g^{ab}} - \rho h_{ab}$, the energy density being written, for convenience, as $\rho = n\epsilon$, ϵ being the energy per particle.

Choosing an orthonormal tetrad $\{u, x, y, z\}$ in M , with u in the direction of the velocity field of the matter and x, y, z spacelike vectors, satisfying the orthogonality conditions $-u_a u^a = x_a x^a = y_a y^a = z_a z^a = 1$, all other inner products being zero, the space-time metric can be written as

$$g_{ab} = -u_a u_b + h_{ab} = -u_a u_b + x_a x_b + y_a y_b + z_a z_b. \quad (4)$$

Here we will choose the spacelike vectors of the tetrad along the eigendirections of $k_b^a = g^{ac}k_{cb}$, so that

$$k_{ab} = n_1^2 x_a x_b + n_2^2 y_a y_b + n_3^2 z_a z_b, \quad (5)$$

where n_1^2, n_2^2 and n_3^2 are the (positive) eigenvalues of k_a^b . The linear particle densities n_1, n_2 and n_3 satisfy $n = n_1 n_2 n_3$. It should be noticed that those eigenvectors are automatically orthogonal whenever the eigenvalues referred above are distinct. However, if the eigenvalues are not all distinct, the eigendirections associated to the same eigenvalue can (and will) be chosen orthogonal.

It is convenient to consider the spatially projected connection D_a acting on an arbitrary tensor field $t_{c\dots}^{b\dots}$ as follows:

$$D_a t_{c\dots}^{b\dots} = h_a^d h_e^b \dots h_c^f \dots \nabla_d t_{f\dots}^{e\dots}. \quad (6)$$

Here ∇ is the connection associated with g and one has $D_a h_{bc} = 0$. Another operator \tilde{D} , such that its action on the same tensor is

$$\tilde{D}_a t_{c\dots}^{b\dots} = h_a^d h_e^b \dots h_c^f \dots \tilde{\nabla}_d t_{f\dots}^{e\dots} \quad (7)$$

is also considered. One has

$$\tilde{D}_b X^a = D_b X^a + S^a_{bc} X^c, \quad (8)$$

for any space-time vector field X . The tensor field S^a_{bc} is the elasticity difference tensor introduced by Karlovini and Samuelsson in [1]. This third order tensor can be written as

$$S^a_{bc} = \frac{1}{2}k^{-am}(D_b k_{mc} + D_c k_{mb} - D_m k_{bc}), \quad (9)$$

where k^{-am} is such that $k^{-am}k_{mb} = h^a_b$. This tensor is used by the same authors to write the Hadamard elasticity tensor, used to describe elasticity properties in space-time, and the Euler equations $\nabla_b T^{ab} = 0$ for elastic matter.

The covariant derivative of the timelike unit vector field u can be decomposed as follows

$$u_{a;b} = -\dot{u}_a u_b + D_b u_a = -\dot{u}_a u_b + \frac{1}{3}\Theta h_{ab} + \sigma_{ab} + \omega_{ab}, \quad (10)$$

where \dot{u}_α is the acceleration, $\sigma_{\alpha\beta}$, the symmetric tracefree rate of shear tensor field, $\omega_{\alpha\beta}$, the antisymmetric vorticity tensor field and Θ , the expansion scalar field for the congruence associated with u .

3. PROPERTIES OF THE ELASTICITY DIFFERENCE TENSOR

Here we will investigate the algebraic properties of the elasticity difference tensor. This tensor, important when studying elasticity within the framework of general relativity, is related to the connection of the space-time, as shown in the previous section. The following two properties of the elasticity difference tensor are straightforward:

- (i) it is symmetric in the two covariant indices, i. e.

$$S^a_{bc} = S^a_{cb}; \quad (11)$$

- (ii) it is a completely flowline orthogonal tensor field, i.e.

$$S^a_{bc} u_a = 0 = S^a_{bc} u^b = S^a_{bc} u^c. \quad (12)$$

The elasticity difference tensor can be approached using the space-time connection, as will be shown here.

It is a well known result that the difference between two connections $\tilde{\nabla}$ and ∇ , associated with two different metrics \tilde{g} and g , respectively, defined on U , is the following (1, 2) tensor:

$$C^m_{ml} = \tilde{\Gamma}^n_{ml} - \Gamma^n_{ml}, \quad (13)$$

$\tilde{\Gamma}^n_{ml}$ and Γ^n_{ml} being the Christoffel symbols associated with those two metrics. In a local chart, this tensor can be written as ([29], [30])

$$C^m_{ml} = \frac{1}{2}\tilde{g}^{np}(\tilde{g}_{pm;l} + \tilde{g}_{pl;m} - \tilde{g}_{ml;p}), \quad (14)$$

where \tilde{g}^{np} is such that $\tilde{g}^{np}\tilde{g}_{pr} = \delta^n_r$ and a semi-colon ; represents the covariant derivative with respect to g . The difference tensor C^m_{ml} can be used to write the difference of the Riemann and the Ricci tensors associated with the two metrics in the following form (see e.g. [31]):

$$\tilde{R}^a_{bcd} - R^a_{bcd} = -C^a_{bd;c} + C^a_{bc;d} - C^a_{lc}C^l_{bd} + C^a_{ld}C^l_{bc} \quad (15)$$

and

$$\tilde{R}_{bd} - R_{bd} = -C^a_{bd;a} + C^a_{ba;d} - C^a_{la}C^l_{bd} + C^a_{ld}C^l_{ba}. \quad (16)$$

The projection of the difference tensor orthogonally to u is defined by the expression

$$h_n^a h_b^m h_c^l C_{ml}^n. \quad (17)$$

When the connections used to define the difference tensor are associated with the metrics $g_{ab} = -u_a u_b + h_{ab}$ and $\tilde{g}_{ab} = -u_a u_b + k_{ab}$, then the corresponding difference tensor, projected according to (17), yields (9) i.e. the *elasticity difference tensor* defined in the previous section.

Under this approach, the elasticity difference tensor is the projection, orthogonal to u , of the difference between two connections, one associated with the space-time metric and the other with the metric $\tilde{g}_{ab} = -u_a u_b + k_{ab}$, where k_{ab} is the pull-back of the material metric k_{AB} .

Calculating the spatially projected versions of equation (15), using (6) and (17), yields the following expression for the difference of the Riemann tensors:

$$\begin{aligned} & h_m^f h_g^n h_e^p h_h^q [h_a^m h_n^b h_p^c h_q^d (\tilde{R}_{bcd}^a - R_{bcd}^a)] \\ &= -D_e S_{gh}^f + D_h S_{ge}^f - S_{ke}^f S_{gh}^k + S_{kh}^f S_{ge}^k. \end{aligned} \quad (18)$$

The spatially projection of (16), the difference of the Ricci tensors, can be obtained analogously by equating the indices $a = c$ in the last expression.

Therefore, these expressions, which contain the elasticity difference tensor, give the difference between the Riemann and Ricci tensors associated with the metrics referred to above.

Now we will obtain the tetrad components of the elasticity difference tensor. To do so, it is more convenient to use the following notation for the orthonormal tetrad: $e_\mu^a = (e_0^a, e_1^a, e_2^a, e_3^a) = (u^a, x^a, y^a, z^a)$. Tetrad indices will be represented by greek letters from the second half or the first half of the alphabet according to their variation as follows: $\mu, \nu, \rho \dots = 0 - 3$ and $\alpha, \beta, \gamma \dots = 1 - 3$. The Einstein summation convention and the notation for the symmetric part of tensors will only be applied to coordinate indices, unless otherwise stated. The operation of raising and lowering tetrad indices will be performed with $\eta_{\mu\nu} = \eta^{\mu\nu} = \text{diag}(-1, 1, 1, 1)$ and one has $g_{ab} = e_{\mu a} e_{\nu b} \eta^{\mu\nu}$.

Writing the Ricci rotation coefficients as $\gamma_{\mu\nu\rho} = e_{\mu a; b} e_\nu^a e_\rho^b$, the tetrad components of the elasticity difference tensor can be obtained using the standard relationship

$$S_{\beta\gamma}^\alpha = S_{bc}^a e_a^\alpha e_\beta^b e_\gamma^c \quad (19)$$

the result being

$$\begin{aligned} S_{\beta\gamma}^\alpha &= \frac{1}{2n_\alpha^2} [(n_\alpha^2 - n_\gamma^2) \gamma_{\gamma\beta}^\alpha + (n_\alpha^2 - n_\beta^2) \gamma_{\beta\gamma}^\alpha + (n_\gamma^2 - n_\beta^2) \gamma_{\beta\gamma}^\alpha + D_n(n_\alpha^2) e_\beta^n \delta_\gamma^\alpha \\ &\quad + D_p(n_\alpha^2) e_\gamma^p \delta_\beta^\alpha - D_l(n_\beta^2) e^{l\alpha} \delta_{\beta\gamma}]. \end{aligned} \quad (20)$$

An alternative form for the last expression is:

$$\begin{aligned} S_{\beta\gamma}^\alpha &= \frac{1}{2} [(1 - \epsilon_{\gamma\alpha}) \gamma_{\gamma\beta}^\alpha + (1 - \epsilon_{\beta\alpha}) \gamma_{\beta\gamma}^\alpha + (\epsilon_{\gamma\alpha} - \epsilon_{\beta\alpha}) \gamma_{\beta\gamma}^\alpha + m_{\beta\alpha} \delta_\gamma^\alpha + m_{\gamma\alpha} \delta_\beta^\alpha \\ &\quad - m_{\beta\gamma}^\alpha \delta_{\beta\gamma} \epsilon_{\beta\alpha}], \end{aligned} \quad (21)$$

where $\epsilon_{\gamma\alpha} = \left(\frac{n_\gamma^2}{n_\alpha^2} \right)$ and $m_{\beta\gamma}^\alpha = D_a(\ln n_\beta^2) e^{a\alpha}$.

The Ricci rotation coefficients, when related to the quantities used in the decomposition (10), can be split into the set [32]:

$$\gamma_{0\alpha 0} = \dot{u}_\alpha \quad (22)$$

$$\gamma_{0\alpha\beta} = \frac{1}{3}\Theta\delta_{\alpha\beta} + \sigma_{\alpha\beta} - \epsilon_{\alpha\beta\gamma}\omega^\gamma \quad (23)$$

$$\gamma_{\alpha\beta 0} = -\epsilon_{\alpha\beta\gamma}\Omega^\gamma \quad (24)$$

$$\gamma_{\alpha\beta\gamma} = -A_\alpha\delta_{\beta\gamma} + A_\beta\delta_{\alpha\gamma} - \frac{1}{2}(\epsilon_{\gamma\delta\alpha}N^\delta_\beta - \epsilon_{\gamma\delta\beta}N^\delta_\alpha + \epsilon_{\alpha\beta\delta}N^\delta_\gamma). \quad (25)$$

The quantities A and N appear in the decomposition of the spatial commutation functions $\Gamma^\alpha_{\beta\gamma} = \gamma^\alpha_{\gamma\beta} - \gamma^\alpha_{\beta\gamma}$, given in [33], where N is a symmetric object.

The elasticity difference tensor can be expressed using three second order symmetric tensors, here designated as $M_{bc,\alpha}$, $\alpha = 1, 2, 3$, as follows:

$$S^a_{bc} = M_{bc,\alpha}x^a + M_{bc,\alpha}y^a + M_{bc,\alpha}z^a = \sum_{\alpha=1}^3 M_{bc,\alpha}e^a_\alpha. \quad (26)$$

Here we will study some properties of the three tensors $M_{bc,\alpha}$ in order to understand until which extent the principal directions of the pulled back material metric remain privileged directions of the elasticity difference tensor, i.e. of the tensors $M_{bc,\alpha}$, by studying the eigenvalue-eigenvector problem for these second order tensors.

First, we will obtain a general expression for $M_{bc,\alpha}$, $\alpha = 1, 2, 3$, which depends explicitly on the orthonormal tetrad vectors, the Ricci rotation coefficients and the linear particle densities n_α . This comes from the contraction of S^a_{bc} in (9) with each one of the spatial tetrad vectors, followed by the use of the relationships (5), (6) and appropriate simplifications. The final result is

$$\begin{aligned} M_{bc,\alpha} = & u^m(e_{\alpha m;(b}u_{c)} + u_{(b}e_{\alpha c);m}) + e_{\alpha(b;c)} - e^m_\alpha e_{\alpha(c}e_{\alpha b);m} \\ & + \gamma_{0\alpha\alpha} u_{(b}e_{\alpha c)} - \gamma_{0\alpha 0} u_b u_c \\ & + \frac{1}{n_\alpha}[2n_{\alpha,(b}e_{\alpha c)} + 2n_{\alpha,m}u^m u_{(b}e_{\alpha c)} + n_{\alpha,m}e^m_\alpha e_{\alpha b}e_{\alpha c}] \\ & + \frac{1}{n_\alpha^2}\{-e^m_\alpha(e_{\beta b}e_{\beta c}n_\beta n_{\beta,m} + e_{\gamma b}e_{\gamma c}n_\gamma n_{\gamma,m}) \\ & + n_\gamma^2[(\gamma_{0\gamma\alpha} - \gamma_{\alpha\gamma 0})u_{(b}e_{\gamma c)} + e^m_\alpha(e_{\gamma m;(b}e_{\gamma c)} - e_{\gamma(b}e_{\gamma c);m})] \\ & + n_\beta^2[(\gamma_{0\beta\alpha} - \gamma_{\alpha\beta 0})u_{(b}e_{\beta c)} + e^m_\alpha(e_{\beta m;(b}e_{\beta c)} - e_{\beta(b}e_{\beta c);m})]\}, \end{aligned} \quad (27)$$

where $\gamma \neq \beta \neq \alpha$, for one pair (β, γ) , a comma being used for partial derivatives. It should be noticed that this expression also contains the non-spatial Ricci rotation coefficients given in (22), (23) and (24).

Naturally, the expressions obtained for $M_{bc,\alpha}$ still satisfy the conditions $M_{bc,\alpha}u^b = 0$.

The eigenvalue-eigenvector problem for $M_{bc,\alpha}$ is quite difficult to solve in general. However, one can investigate the conditions for the tetrad vectors to be eigenvectors of those tensors, the results being summarized in the two following theorems.

Intrinsic derivatives of arbitrary scalar fields Φ , as derivatives along tetrad vectors, will be represented Δ_{e_α} and defined as:

$$\Delta_{e_\alpha} \Phi = \Phi_{,m} e_\alpha^m,$$

where a comma is used for partial derivatives.

Theorem 1. *The tetrad vector e_α is an eigenvector for M_α iff n_α remains invariant along the two spatial tetrad vectors e_β , such that $\beta \neq \alpha$ i.e. $\Delta_{e_\beta}(\ln n_\alpha) = 0$ whenever $\beta \neq \alpha$.*

The corresponding eigenvalue is $\lambda = \Delta_{e_\alpha}(\ln n_\alpha)$.

Proof: In order to solve this eigenvector-eigenvalue equation the following algebraic conditions are used

$$M_\alpha^c e_\alpha^b e_{\alpha c} = \lambda, \quad (28)$$

$$M_\alpha^c e_\alpha^b e_{\beta c} = 0 \quad (29)$$

and

$$M_\alpha^c e_\alpha^b e_{\gamma c} = 0, \quad (30)$$

where $\gamma \neq \beta \neq \alpha$. Using the orthogonality conditions satisfied by the tetrad vectors and the properties of the rotation coefficients, namely the fact that they are anti-symmetric on the first pair of indices, (29) and (30) yield $\Delta_{e_\beta}(\ln n_\alpha) = 0 = \Delta_{e_\gamma}(\ln n_\alpha)$ so that $\Delta_{e_\beta} n_\alpha = 0 = \Delta_{e_\gamma} n_\alpha$. On the other hand from (28) one obtains $\lambda = \Delta_{e_\alpha}(\ln n_\alpha)$.

It should be noticed that $\lambda = 0$ whenever n_α remains constant along e_α . However this condition is equivalent to $n_\alpha = c$, with c a constant. In this case, $k_{ab} = c^2 e_{\alpha a} e_{\alpha b} + \sum_{\beta \neq \alpha} n_\beta^2 e_{\beta a} e_{\beta b}$. \square

Theorem 2. *e_β is an eigenvector of M_α iff the following conditions are satisfied:*

- (i) $\Delta_{e_\beta}(\ln n_\alpha) = 0$, i.e. n_α remains invariant along the direction of e_β ;
- (ii) $\gamma_{\alpha\gamma\beta}[n_\alpha^2 - n_\gamma^2] + \gamma_{\alpha\beta\gamma}[n_\alpha^2 - n_\beta^2] + \gamma_{\beta\gamma\alpha}[n_\gamma^2 - n_\beta^2] = 0$, where $\gamma \neq \beta \neq \alpha$ for one pair (β, γ) .

The corresponding eigenvalue is $\lambda = -\frac{n_\beta}{n_\alpha^2} \Delta_{e_\alpha} n_\beta + \gamma_{\alpha\beta\beta}(-\frac{n_\beta}{n_\alpha^2} + 1)$.

Proof: Contracting $M_\alpha^c e_\beta^b = \lambda e_\beta^c$ with $e_{\alpha c}$ one obtains $\Delta_{e_\beta}(\ln n_\alpha) = 0$. This condition is satisfied whenever $\Delta_{e_\beta} n_\alpha = 0$. The second condition results from $M_\alpha^c e_\beta^b e_{\gamma c} = 0$. And contracting $M_\alpha^c e_\beta^b = \lambda e_\beta^c$ with $e_{\beta c}$ yields the eigenvalue λ . The used simplifications are based on the orthogonality conditions of the tetrad vectors and on the properties of the rotation coefficients.

Notice that the two conditions are satisfied if $n_\alpha = n_\beta = n_\gamma = c$, where c is a constant. The consequence of this is that $\lambda = 0$. In this case, $k_{ab} = c^2 x_a x_b + c^2 y_a y_b + c^2 z_a z_b$. \square

The previous theorems show that strong conditions have to be imposed on n_α , for $\alpha = 1, 2, 3$, and the metric in order that the spatial tetrad vectors are principal directions of M_α , for $\alpha = 1, 2, 3$.

However, the conditions to have e_α as eigenvector of M_α seem less restrictive than the conditions for e_β , for all values of $\beta \neq \alpha$, to be eigenvector of the same tensor M_α , since these involve not only intrinsic derivatives of the scalar fields but also rotation coefficients of the metric. Furthermore, for e_α to be an eigenvector of M_α only conditions on n_α have to be satisfied, namely that n_α remains constant along the directions of e_β for all values of $\beta \neq \alpha$ (in which case the eigenvalue corresponding to e_α depends only on n_α). On the other hand, the conditions imposed for e_β for all $\beta \neq \alpha$ to be eigenvectors of M_α also involve n_β for all $\beta \neq \alpha$.

Next we will use the previous theorems to establish the conditions for e_α , with $\alpha = 1, 2, 3$ to be an eigenvector of the three tensors M_1, M_2, M_3 simultaneously, the results being:

$$\begin{aligned} \text{(i)} \quad & \Delta_{e_\beta}(\ln n_\alpha) = 0, \\ \text{(ii)} \quad & \Delta_{e_\alpha}(\ln n_\beta) = 0, \\ \text{(iii)} \quad & \gamma_{\alpha\beta\gamma} [n_\alpha^2 - n_\beta^2] + \gamma_{\alpha\gamma\beta} [n_\gamma^2 - n_\alpha^2] + \gamma_{\beta\gamma\alpha} [n_\beta^2 - n_\gamma^2] = 0, \end{aligned}$$

for all values of β and γ such that $\beta \neq \gamma \neq \alpha$.

These conditions must be satisfied for all values of $\beta \neq \alpha$. It is not easy to find the general solution to these equations, however one can say that, in general, the principal directions of the pulled back material metric k are not, in general, the principal directions of the three tensors M_1, M_2 and M_3 . It should be noticed that the (mathematical) solution corresponding to $n_1 = n_2 = n_3 = \text{const.}$ is not an interesting result from the physical point of view.

As a special case, we now consider that all eigenvalues of k^a_b are equal, i.e.

$$n_1 = n_2 = n_3 = n^{\frac{1}{3}}. \quad (31)$$

Therefore, $k_{ab} = n^{\frac{2}{3}} h_{ab}$, so that these tensors are conformally related. In physical terms, this corresponds to the unsheared state described in [1]: the energy per particle, ϵ , has a minimum under variations of g^{AB} such that n is held fixed. The above theorems in this section simplify significantly in this case, as can easily be proved using (31) in those theorems. For completeness, we give the expressions for the elasticity difference tensor and the tensors M_α in this special case:

$$\begin{aligned} S^a_{bc} &= \frac{1}{3} \frac{1}{n} (\delta^a_c D_b n + \delta^a_b D_c n - h^{ad} h_{bc} D_d n), \\ M_{bc} &= \frac{1}{3} \frac{1}{n} (e_{\alpha c} n_{,b} + e_{\alpha b} n_{,c} + (e_{\alpha c} u_b + e_{\alpha b} u_c) \Delta_{e_0} n - h_{bc} \Delta_{e_\alpha} n). \end{aligned}$$

4. EXAMPLES

Here, examples concerning the static spherically symmetric case and an axially symmetric, non-rotating metric are presented, where we apply the analysis developed in the last section. The main problem when dealing with examples lies in the difficulties of finding an orthonormal tetrad for the space-time metric such that the corresponding spacelike vectors are precisely the principal directions of the pulled back material metric. However, in the examples presented, this difficulty was overcome.

4.1. The static spherically symmetric case. In this section we analyse the elasticity difference tensor and corresponding eigendirections for the static spherically symmetric metric, due to its significance on modelling neutron stars. The metric regarded here can be thought of as the interior metric of a non rotating star composed by an elastic material.

For a static spherically symmetric spacetime the line-element can be written as

$$ds^2 = -e^{2\nu(r)}dt^2 + e^{2\lambda(r)}dr^2 + r^2d\theta^2 + r^2\sin^2\theta d\phi^2, \quad (32)$$

where the coordinates $\omega^a = \{t, r, \theta, \phi\}$ are, respectively, the time coordinate, the radial coordinate, the axial coordinate and the azimuthal coordinate. Choosing the basis one-forms $u_a = (-e^{\nu(r)}, 0, 0, 0)$, $x_a = (0, e^{\lambda(r)}, 0, 0)$, $y_a = (0, 0, r, 0)$ and $z_a = (0, 0, 0, r\sin\theta)$ for the orthonormal tetrad, the metric is given by $g_{ab} = -u_a u_b + x_a x_b + y_a y_b + z_a z_b$ and $h_{ab} = x_a x_b + y_a y_b + z_a z_b$ defines the corresponding projection tensor. Using this tetrad, the pulled-back material metric becomes

$$k_{ab} = n_1^2 x_a x_b + n_2^2 y_a y_b + n_2^2 z_a z_b, \quad (33)$$

where we have chosen $n_3 = n_2$ since for this material distribution k has only two different eigenvalues.

Let $\xi^A = \{\tilde{r}, \tilde{\theta}, \tilde{\phi}\}$ be the coordinate system in the material space \mathcal{X} . Since the space-time is static and spherically symmetric, \tilde{r} can only depend on r and one can take $\tilde{\theta} = \theta$ and $\tilde{\phi} = \phi$ so that the configuration of the material is entirely described by the material radius $\tilde{r}(r)$. Moreover, the only non-zero components of the deformation gradient are $\frac{d\xi^1}{d\omega^1} = \frac{d\tilde{r}}{dr}$, $\frac{d\xi^2}{d\omega^2} = 1$ and $\frac{d\xi^3}{d\omega^3} = 1$.

In \mathcal{X} the material metric is $k_{AB} = \tilde{x}_A \tilde{x}_B + \tilde{y}_A \tilde{y}_B + \tilde{z}_A \tilde{z}_B$, with $\tilde{x}_A = e^{\tilde{\lambda}} d\tilde{r}_A$, $\tilde{y}_A = \tilde{r} d\tilde{\theta}_A$ and $\tilde{z}_A = \tilde{r} \sin\tilde{\theta} d\tilde{\phi}_A$, and where $\tilde{\lambda} = \lambda(\tilde{r})$. The pull-back of the material metric is then

$$k^a{}_b = g^{ac} k_{cb} = g^{ac} (\xi^C \xi_b^B k_{CB}) = \left(\frac{d\tilde{r}}{dr} \right)^2 e^{2\tilde{\lambda}-2\lambda} \delta_1^a \delta_b^1 + \frac{\tilde{r}^2}{r^2} \delta_2^a \delta_b^2 + \frac{\tilde{r}^2}{r^2} \delta_3^a \delta_b^3, \quad (34)$$

Comparing (33) and (34) it is simple to obtain the following values for the linear particle densities (all positive), which are found to depend on r only:

$$n_1 = n_1(r) = \frac{d\tilde{r}}{dr} e^{\tilde{\lambda}-\lambda} \quad (35)$$

$$n_2 = n_2(r) = n_3(r) = \frac{\tilde{r}}{r} \quad (36)$$

The non-zero components of the strain tensor (3), when written as functions of the quantities n_α , are

$$\begin{aligned} s_{rr} &= \frac{1}{2} e^{2\lambda} (1 - n^{-\frac{2}{3}} n_1^2) \\ s_{\theta\theta} &= \frac{1}{2} r^2 (1 - n^{-\frac{2}{3}} n_2^2) \\ s_{\phi\phi} &= \frac{1}{2} r^2 \sin^2\theta (1 - n^{-\frac{2}{3}} n_2^2) \end{aligned}$$

Using the expressions obtained for the n_α one can find that the condition for this tensor to vanish identically is that $\tilde{r} = ce^{\pm \int \frac{1}{\tilde{r}} e^{\lambda-\tilde{\lambda}} dr}$.

Calculating the quantities given in (10) one obtains

$$\begin{aligned}\Theta &= 0 \\ \dot{u}_a &= \left(0, e^{2\nu} \frac{d\nu}{dr}, 0, 0\right) \\ \sigma_{ab} : \quad \sigma_{12} &= \frac{1}{2} e^{4\nu} \frac{d\nu}{dr} = \sigma_{21} \\ \omega_{ab} : \quad \omega_{12} &= e^{2\nu} \frac{d\nu}{dr} + \frac{1}{2} e^{4\nu} \frac{d\nu}{dr} \\ \omega_{21} &= -\omega_{12},\end{aligned}$$

where the remaining components of σ_{ab} and ω_{ab} vanish.

The non-zero components of the elasticity difference tensor S_{bc}^a are:

$$\begin{aligned}S_{rr}^r &= \frac{1}{n_1} \frac{dn_1}{dr} \\ S_{\theta r}^\theta &= \frac{1}{n_2} \frac{dn_2}{dr} \\ S_{\phi r}^\phi &= \frac{1}{n_2} \frac{dn_2}{dr} \\ S_{\theta\theta}^r &= r e^{-2\lambda} - r e^{-2\lambda} \frac{n_2^2}{n_1^2} - e^{-2\lambda} r^2 \frac{n_2}{n_1^2} \frac{dn_2}{dr} \\ S_{\phi\phi}^r &= e^{-2\lambda} r \sin^2 \theta - e^{-2\lambda} r \sin^2 \theta \frac{n_2^2}{n_1^2} - e^{-2\lambda} r^2 \sin^2 \theta \frac{n_2}{n_1^2} \frac{dn_2}{dr}.\end{aligned}$$

Since $S_{bc}^a = S_{cb}^a$, there are only seven non-zero components for this tensor on the coordinate system chosen above.

Again, using (35) and (36) one obtains that:

- (i) the components $S_{\theta r}^\theta$ and $S_{\phi r}^\phi$ are zero whenever the function \tilde{r} is of the form $\tilde{r} = c_1 r$, where c_1 is a constant;
- (ii) S_{rr}^r is zero whenever $\tilde{r} = c_2 + c_3 \int e^{\lambda - \tilde{\lambda}} dr$;
- (iii) the components $S_{\theta\theta}^r$ and $S_{\phi\phi}^r$ are zero whenever $\tilde{r} = c_4 e^{\int \frac{\epsilon - 2\tilde{\lambda} + 2\lambda}{r} dr}$.

The second order symmetric tensors M_α , for $\alpha = 1, 2, 3$ have the following non-zero components:

$$\begin{aligned}M_{rr}^1 &= \frac{e^\lambda}{n_1} \frac{dn_1}{dr} \\ M_{\theta\theta}^1 &= e^{-\lambda} r - e^{-\lambda} r \frac{n_2^2}{n_1^2} - e^{-\lambda} r^2 \frac{n_2}{n_1^2} \frac{dn_2}{dr} \\ M_{\phi\phi}^1 &= e^{-\lambda} r \sin^2 \theta - e^{-\lambda} r \sin^2 \theta \frac{n_2^2}{n_1^2} - e^{-\lambda} r^2 \sin^2 \theta \frac{n_2}{n_1^2} \frac{dn_2}{dr} \\ M_{r\theta}^2 &= M_{\theta r}^2 = \frac{r}{n_2} \frac{dn_2}{dr} \\ M_{r\phi}^3 &= M_{\phi r}^3 = \frac{r \sin \theta}{n_2} \frac{dn_2}{dr}\end{aligned}$$

The eigenvalues and eigenvectors of these tensors are presented in tables 1, 2 and 3, being then compared with the eigendirections of the material metric.

Table 1 - Eigenvectors and eigenvalues for M_1

Eigenvectors	Eigenvalues
x	$\mu_1 = \frac{e^{-\lambda}}{n_1} \frac{dn_1}{dr}$
y	$\mu_2 = \frac{e^{-\lambda}}{r} - \frac{e^{-\lambda}}{r} \frac{n_2}{n_1^2} - e^{-\lambda} \frac{n_2}{n_1^2} \frac{dn_2}{dr}$
z	$\mu_3 = \frac{e^{-\lambda}}{r} - \frac{e^{-\lambda}}{r} \frac{n_2}{n_1^2} - e^{-\lambda} \frac{n_2}{n_1^2} \frac{dn_2}{dr}$

Notice that, in the present example, M_1 maintains the eigenvectors of k , namely x , y and z , the two last ones being associated with the same eigenvalue. Therefore the canonical form for M_1 is $M_{bc} = \mu_1 x_b x_c + \mu_2 (y_b y_c + z_b z_c)$, where μ_1 and μ_2 are the eigenvalues corresponding to x and y ($\equiv z$), respectively.

Table 2 - Eigenvectors and eigenvalues for M_2

Eigenvectors	Eigenvalues
$x + y$	$\mu_4 = \frac{e^{-\lambda}}{n_2} \frac{dn_2}{dr}$
$x - y$	$\mu_5 = -\frac{e^{-\lambda}}{n_2} \frac{dn_2}{dr}$
z	$\mu_6 = 0$

In this case, only the eigenvector z of k remains as eigenvector, however the corresponding eigenvalue being zero. The other two eigenvectors are $x + y$ and $x - y$ so that the canonical form for M_2 can be expressed as $M_{bc} = 2\mu_4 (x_b y_c + y_b x_c)$, where $\mu_4 = e^{-\lambda} \left(\frac{1}{\tilde{r}} \frac{d\tilde{r}}{dr} - \frac{1}{r} \right)$.

Table 3 - Eigenvectors and eigenvalues of M_3

Eigenvectors	Eigenvalues
$x + z$	$\mu_7 = \frac{e^{-\lambda}}{n_2} \frac{dn_2}{dr}$
$x - z$	$\mu_8 = -\frac{e^{-\lambda}}{n_2} \frac{dn_2}{dr}$
y	$\mu_9 = 0$

Comparing M_2 and M_3 , it is easy to see that the role of z and y is interchanged. The eigenvalues of M_2 are equal to the eigenvalues of M_3 and the canonical form of this tensor field can be written as $M_{bc} = 2\mu_7 (x_b z_c + z_b x_c)$, where $\mu_7 = e^{-\lambda} \left(\frac{1}{\tilde{r}} \frac{d\tilde{r}}{dr} - \frac{1}{r} \right)$.

It should be noticed that the case n_2 constant is not interesting to analyze, since this corresponds to the vanishing of the tensors M_2 and M_3 .

x , y and z would only remain eigenvectors for M_2 and M_3 if \tilde{r} would be of the form: $\tilde{r} = cr$, in which case M_2 and M_3 were reduced to a zero tensor.

The tetrad components of the elasticity difference tensor can directly be obtained from (21):

$$\begin{aligned}
S^1_{11} &= e^{-\lambda} \frac{1}{n_1} \frac{dn_1}{dr} \\
S^2_{21} &= e^{-\lambda} \frac{1}{n_2} \frac{dn_2}{dr} \\
S^3_{31} &= e^{-\lambda} \frac{1}{n_2} \frac{dn_2}{dr} \\
S^1_{22} &= e^{-\lambda} \frac{1}{r} - e^{-\lambda} \frac{1}{r} \frac{n_2^2}{n_1^2} - e^{-\lambda} \frac{n_2}{n_1^2} \frac{dn_2}{dr} \\
S^1_{33} &= e^{-\lambda} \frac{1}{r} - e^{-\lambda} \frac{1}{r} \frac{n_2^2}{n_1^2} - e^{-\lambda} \frac{n_2}{n_1^2} \frac{dn_2}{dr}.
\end{aligned}$$

The expressions for the Ricci rotation coefficients are

$$\begin{aligned}
\gamma_{122} &= \frac{e^{-\lambda}}{r} \\
\gamma_{133} &= \frac{e^{-\lambda}}{r} \\
\gamma_{233} &= \frac{\cos \theta}{r \sin \theta}.
\end{aligned}$$

4.2. The axially symmetric non-rotating case. First, consider an elastic, axially symmetric, uniformly rotating body in interaction with its gravitational field. The exterior of the body may be described by the following metric, [27],

$$ds^2 = -e^{2\nu} dt^2 + e^{2\mu} dr^2 + e^{2\mu} dz^2 + e^{2\psi} (d\phi - \omega dt)^2, \quad (37)$$

where ν, ψ, ω, μ are scalar fields depending on r and z .

Assume that the material metric is flat. Introducing in \mathcal{X} cylindrical coordinates $\xi^A = \{R, \zeta, \Phi\}$, then the material metric takes the form:

$$ds^2 = dR^2 + d\zeta^2 + R^2 d\Phi^2, \quad (38)$$

where the parameters R, ζ depend on r and z , Φ being $\Phi(t, r, z, \phi) = \phi - \Omega(r, z)t$.

Now, consider the limiting case of an axially symmetric non-rotating elastic system for which the space-time metric is given by

$$ds^2 = -e^{2\nu} dt^2 + e^{2\mu} dr^2 + e^{2\mu} dz^2 + e^{2\psi} d\phi^2. \quad (39)$$

This metric is obtained from (37), when $\omega = 0$ and the angular velocity $\Omega = 0$.

Imposing $R = R(r)$, $\zeta = z$ and $g_{ab} = g_{ab}(r)$, one obtains a further reduction to cylindrical symmetry. This reduction is considered in [27].

So, the space-time metric we will work with is given by (39), where ν, μ, ψ depend on r only, and it can be written as $g_{ab} = -u_a u_b + x_a x_b + y_a y_b + z_a z_b$, where $u_a = (-e^{\nu(r)}, 0, 0, 0)$, $x_a = (0, e^{\mu}, 0, 0)$, $y_a = (0, 0, e^{\mu(r)}, 0)$ and $z_a = (0, 0, 0, e^{\psi(r)})$. The space-time coordinates are $\omega^a = \{t, r, z, \phi\}$.

In \mathcal{X} the material metric k_{AB} is given by $k_{AB} = \tilde{x}_A \tilde{x}_B + \tilde{y}_A \tilde{y}_B + \tilde{z}_A \tilde{z}_B$, where $\tilde{x}_A = dR_A$, $\tilde{y}_A = dz_A$ and $\tilde{z}_A = R d\phi_A$. The relativistic deformation gradient has

the following non-zero components $\frac{d\xi^1}{d\omega^2} = \frac{dR}{dr}$, $\frac{d\xi^2}{d\omega^1} = 1$ and $\frac{d\xi^3}{d\omega^3} = 1$. Calculating the pull-back of the material metric one obtains

$$k^a{}_b = g^{ac}k_{cb} = g^{ac}(\xi^C_c \xi^B_b k_{CB}) = e^{-2\mu}\delta^a_1\delta^1_b + \left(\frac{dR}{dr}\right)^2 e^{-2\mu}\delta^a_2\delta^2_b + R^2 e^{-2\psi}\delta^a_3\delta^3_b. \quad (40)$$

The corresponding line-element can be expressed as

$$ds^2 = dr^2 + \left(\frac{dR}{dr}\right)^2 dz^2 + R^2 d\phi^2. \quad (41)$$

On the other hand, the material metric in the space-time M is given by

$$k_{ab} = n_1^2 x_a x_b + n_2^2 y_a y_b + n_3^2 z_a z_b. \quad (42)$$

Comparing (40) with (42) one concludes that the linear particle densities (all positive) are expressed as

$$n_1 = n_1(r) = e^{-\mu} \quad (43)$$

$$n_2 = n_2(r) = e^{-\mu} \frac{dR}{dr} \quad (44)$$

$$n_3 = n_3(r) = R e^{-\psi}. \quad (45)$$

The strain tensor (3) is composed of the following components

$$\begin{aligned} s_{rr} &= \frac{1}{2} e^{2\mu} (1 - n^{-\frac{2}{3}} n_1^2) \\ s_{zz} &= \frac{1}{2} e^{2\mu} (1 - n^{-\frac{2}{3}} n_2^2) \\ s_{\phi\phi} &= \frac{1}{2} e^{2\psi} (1 - n^{-\frac{2}{3}} n_3^2) \end{aligned}$$

The strain tensor vanishes if the condition $R(r) = r = e^{\psi-\mu}$ is satisfied.

Calculating the quantities given in (10) one obtains

$$\begin{aligned} \Theta &= 0 \\ \dot{u}_a &= \left(0, e^{2\nu} \frac{d\nu}{dr}, 0, 0\right) \\ \sigma_{ab} : \quad \sigma_{12} &= \frac{1}{2} e^{4\nu} \frac{d\nu}{dr} = \sigma_{21} \\ \omega_{ab} : \quad \omega_{12} &= e^{2\nu} \frac{d\nu}{dr} + \frac{1}{2} e^{4\nu} \frac{d\nu}{dr} \\ \omega_{21} &= -\omega_{12}, \end{aligned}$$

where the remaining components of σ_{ab} and ω_{ab} vanish.

The non-zero components of the elasticity difference tensor are

$$\begin{aligned}
S^r_{rr} &= \frac{1}{n_1} \frac{dn_1}{dr} \\
S^z_{zr} &= \frac{1}{n_2} \frac{dn_2}{dr} \\
S^\phi_{\phi r} &= \frac{1}{n_3} \frac{dn_3}{dr} \\
S^r_{zz} &= \frac{d\mu}{dr} - \frac{n_2^2}{n_1^2} \frac{d\mu}{dr} - \frac{n_2}{n_1^2} \frac{dn_2}{dr} \\
S^r_{\phi\phi} &= e^{-2\psi-2\mu} \left(\frac{d\psi}{dr} - \frac{n_3^2}{n_1^2} \frac{d\psi}{dr} - \frac{n_3}{n_1^2} \frac{dn_3}{dr} \right).
\end{aligned}$$

It can be observed that only seven components of the elasticity difference tensor are non-zero.

Using the expressions (43), (44) and (45) one can conclude that:

- (i) S^r_{rr} is zero whenever $\mu(r) = c$, where c is a constant;
- (ii) S^z_{zr} is zero whenever $R(r) = c_1 + c_2 \int e^{\mu(r)} dr$;
- (iii) $S^\phi_{\phi r}$ is zero whenever $R(r) = c_3 e^{\psi(r)}$;
- (iv) S^r_{zz} is zero whenever $R(r) = \pm \int \sqrt{2\mu(r) + c_4} dr + c_5$;
- (v) $S^r_{\phi\phi}$ is zero whenever $R(r) = \pm \sqrt{2 \int \frac{e^{2\psi}}{e^{2\mu}} \frac{d\psi}{dr} dr} + c_6$.

The second-order tensors M_1 , M_2 and M_3 have the following non-zero components:

$$\begin{aligned}
M_{rr}^1 &= e^\mu \frac{1}{n_1} \frac{dn_1}{dr} \\
M_{zz}^1 &= e^\mu \left(\frac{d\mu}{dr} - \frac{n_2^2}{n_1^2} \frac{d\mu}{dr} - \frac{n_2}{n_1^2} \frac{dn_2}{dr} \right) \\
M_{\phi\phi}^1 &= e^{2\psi-\mu} \left(\frac{d\psi}{dr} - \frac{n_3^2}{n_1^2} \frac{d\psi}{dr} - \frac{n_3}{n_1^2} \frac{dn_3}{dr} \right) \\
M_{rz}^2 &= M_{zr}^2 = e^\mu \frac{1}{n_2} \frac{dn_2}{dr} \\
M_{r\phi}^3 &= M_{\phi r}^3 = e^\psi \frac{1}{n_3} \frac{dn_3}{dr}.
\end{aligned}$$

The next three tables contain the eigenvalues and eigenvectors for these tensors, which are then compared with the eigenvectors of the pulled-back material metric.

Table 1 - Eigenvectors and eigenvalues for M_1

Eigenvectors	Eigenvalues
x	$\lambda_1 = e^{-\mu} \frac{1}{n_1} \frac{dn_1}{dr}$
y	$\lambda_2 = e^{-\mu} \left(\frac{d\mu}{dr} - \frac{n_2^2}{n_1^2} \frac{d\mu}{dr} - \frac{n_2}{n_1^2} \frac{dn_2}{dr} \right)$
z	$\lambda_3 = e^{-\mu} \left(\frac{d\psi}{dr} - \frac{n_3^2}{n_1^2} \frac{d\psi}{dr} - \frac{n_3}{n_1^2} \frac{dn_3}{dr} \right)$

One can observe that the eigendirections x , y and z of k are also eigenvectors for the tensor M_1 and the eigenvectors are associated with different eigenvalues. The canonical form for M_1 can be written as $M_{bc} = \lambda_1 x_b x_c + \lambda_2 y_b y_c + \lambda_3 z_b z_c$.

Table 2 - Eigenvectors and eigenvalues for M_2

Eigenvectors	Eigenvalues
$x + y$	$\lambda_4 = e^{-\mu} \left(\frac{1}{n_2} \frac{dn_2}{dr} \right)$
$x - y$	$\lambda_5 = -e^{-\mu} \left(\frac{1}{n_2} \frac{dn_2}{dr} \right)$
z	$\lambda_6 = 0$

M_2 inherits only the eigenvector z of k , which corresponds to a zero eigenvalue. The other two eigenvectors of M_2 are linear combinations of x and y : $x + y$ and $x - y$, whose corresponding eigenvalues are symmetric in sign. The canonical form for M_2

can be written as $M_{bc} = 2\lambda_4(x_b y_c + y_b x_c)$, where $\lambda_4 = \left(\frac{\frac{d^2 R}{dr^2}}{\frac{dR}{dr}} - \frac{d\mu}{dr} \right) e^{-\mu}$.

Table 3 - Eigenvectors and eigenvalues for M_3

Eigenvectors	Eigenvalues
$x + z$	$\lambda_7 = e^{-\mu} \left(\frac{1}{n_3} \frac{dn_3}{dr} \right)$
$x - z$	$\lambda_8 = -e^{-\mu} \left(\frac{1}{n_3} \frac{dn_3}{dr} \right)$
y	$\lambda_9 = 0$

M_3 inherits the eigenvalue y of k , which is associated with the eigenvalue zero. The other two eigenvectors of M_3 are linear combinations of x and z : $x + z$ and $x - z$. These two eigenvectors are associated with sign symmetric eigenvalues. The canonical form for M_3 can be written as $M_{bc} = 2\lambda_7(x_b z_c + z_b x_c)$, where $\lambda_7 = \left(\frac{1}{R} \frac{dR}{dr} - \frac{d\psi}{dr} \right) e^{-\mu}$.

x and y would only be eigenvectors for M_2 if $R(r)$ would be of the form $R(r) = c_1 + \int e^\mu dr c_2$, but in this case M_2 would vanish. x and z would only be eigenvectors for M_3 if $R(r)$ would be of the form $R(r) = c_3 e^\psi$ and this would reduce M_3 to a zero tensor.

One can observe that the role that y and n_2 play for the tensor M_2 is the same that z and n_3 play for M_3 . That is, the results for M_2 and M_3 are very similar, only y and n_2 are substituted by z and n_3 , respectively.

The tetrad components of the elasticity difference tensor obtained from (21) and the expressions for the Ricci rotation coefficients are listed below:

$$\begin{aligned}
S^1_{11} &= e^{-\mu} \frac{1}{n_1} \frac{dn_1}{dr} \\
S^2_{21} &= e^{-\mu} \frac{1}{n_2} \frac{dn_2}{dr} \\
S^3_{31} &= e^{-\mu} \frac{1}{n_3} \frac{dn_3}{dr} \\
S^1_{22} &= e^{-\mu} \frac{d\mu}{dr} - e^{-\mu} \frac{n_2^2}{n_1^2} \frac{d\mu}{dr} - e^{-\mu} \frac{n_2}{n_1^2} \frac{dn_2}{dr} \\
S^1_{33} &= e^{-\mu} \frac{d\psi}{dr} - e^{-\mu} \frac{n_3^2}{n_1^2} \frac{d\psi}{dr} - e^{-\mu} \frac{n_3}{n_1^2} \frac{dn_3}{dr}.
\end{aligned}$$

The expressions for the Ricci coefficients are

$$\gamma_{122} = \frac{\frac{d\mu}{dr}}{e^\mu}$$

$$\gamma_{133} = \frac{\frac{d\psi}{dr}}{e^\mu}.$$

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